THE PROBLEM OF HEAT CONDUCTION IN AN INFINITELY LONG PERFORATED STRIP

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A formula is derived for the temperature distribution in a strip with uniformly spaced holes.

We consider an infinitely long strip of width $2a$, with an infinitely long row of uniformly spaced circular holes of radius R. The center-to-center distance between the holes is 2b. The centerline of the holes is offset from the centerline of the strip by a distance e (Fig. 1).

We will determine the temperature distribution T in the strip, if the outer lateral surface is held at temperature T = 0 and the edges of the holes are held at temperature T = M. The strip based surfaces are thermally insulated. We assume that the thermal properties of the strip material do not depend on the temperature.

As long as the holes are uniformly spaced, we can single out a rectangular strip segment ABCD and analyze its thermal state. In x, y coordinates we have the following boundary conditions at the sides of this rectangle:

$$T = 0 \quad \text{at} \quad x = a - e \ \text{and} \ x = -a - e,$$

$$\frac{\partial T}{\partial y} = 0 \quad \text{at} \quad y = \pm b.$$

We will now change to polar coordinates in dimensionless variables. The differential equation of heat conduction and the boundary conditions are [1]:

$$\frac{\partial^2\theta}{\partial r^2} + \frac{1}{r}\frac{\partial\theta}{\partial r} + \frac{1}{r^2}\frac{\partial^2\theta}{\partial\varphi^2} = 0, \tag{1}$$

$$\theta = 0 \quad \text{at} \quad \begin{cases} r = \dfrac{1-\beta}{\cos\varphi}, & 0 \leqslant \varphi \leqslant \varphi_1; \\ r = -\dfrac{1+\beta}{\cos\varphi}, & \varphi_2 \leqslant \varphi \leqslant \pi; \end{cases} \tag{2}$$

$$\sin\varphi \frac{\partial\theta}{\partial r} + \frac{1}{r}\cos\varphi\frac{\partial\theta}{\partial\varphi} = 0 \quad \text{at} \quad r = \frac{\gamma}{\sin\varphi}, \ \varphi_1 \leqslant \varphi \leqslant \varphi_2; \tag{3}$$

$$\theta = 1 \quad \text{at} \quad r = \alpha, \ 0 \leqslant \varphi \leqslant \pi. \tag{4}$$

We introduce the following symbols:

$$\theta = \frac{T}{M}, \quad \alpha = \frac{R}{a}, \quad \beta = \frac{e}{a}, \quad \gamma = \frac{b}{a},$$

$$\varphi_1 = \operatorname{arc\,tg}\frac{\gamma}{1-\beta}, \quad \varphi_2 = \pi - \operatorname{arc\,tg}\frac{\gamma}{1+\beta}.$$

The solution to Eq. (1) is sought in the form of a polynomial:

$$\theta = P_0 + Q_0 \ln r - \sum_{n=1}^{k}\left(Q_n r^n + \frac{P_n}{r^n}\right)\cos n\varphi, \tag{5}$$

with the constants P_0, Q_0, P_n, and Q_n determined from the boundary conditions.

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TABLE 1. Values of the Coefficients when the Centerline of the Holes Is Not Offset from the Centerline of the Strip (i.e., $\beta = 0$ and $\gamma = 1$)

α	Q_0	$Q_2 \cdot 10$	$Q_4 \cdot 10^2$	$Q_6 \cdot 10^3$	$Q_8 \cdot 10^3$	$Q_{10} \cdot 10^3$	$Q_{12} \cdot 10^4$
0,05	0,2926	1,2594	—0,5008	0,2771	—	—	—
0,10	0,3671	1,5797	—0,6280	0,3476	—	—	—
0,15	0,4312	1,8560	—0,7382	0,4084	—	—	—
0,20	0,4924	2,1185	—0,8368	0,4660	—	—	—
0,25	0,5535	2,3795	—0,9306	0,5230	—	—	—
0,30	0,6162	2,6457	—1,0160	0,5808	—	—	—
0,35	0,6820	2,9224	—1,0883	0,6401	—	—	—
0,40	0,7524	3,2142	—1,1399	0,7015	—	—	—
0,45	0,8293	3,5264	—1,1603	0,7653	—	—	—
0,50	0,9151	3,8748	—1,0064	0,8540	0,2372	—	—
0,55	1,0131	4,2626	—0,7357	0,9549	0,5636	—	—
0,60	1,1278	4,7070	—0,2736	1,0773	1,0595	—	—
0,65	1,2662	5,2314	0,4657	1,2322	1,7951	—	—
0,70	1,4390	5,8699	1,5528	1,3410	2,4812	0,2227	—
0,75	1,6650	6,7085	3,7058	2,0206	7,4575	0,6756	4,1121
0,80	1,9764	7,8522	6,6759	2,6530	11,3152	1,0466	6,1971
0,85	2,4420	9,5891	11,7081	3,8945	17,7711	1,9417	9,4755

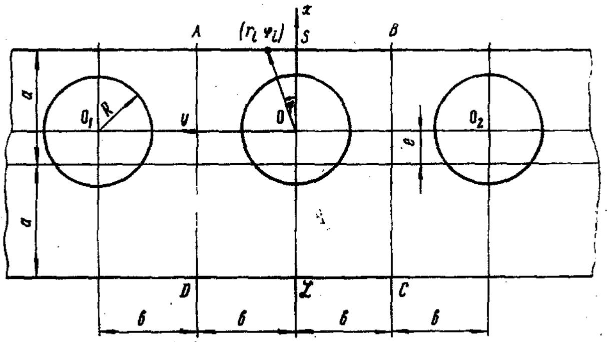

Fig. 1. Strip with uniformly spaced holes.

From the condition at the inner contour (4) follows

$$\theta = 1 + Q_0 \ln \frac{r}{\alpha} - \sum_{n=1}^{k} \left[r^n - \left(\frac{\alpha^2}{r} \right)^n \right] Q_n \cos n\varphi. \quad (6)$$

With the aid of formula (6), we can express conditions (2) and (3) at the outer contour respectively as

$$1 + Q_0 \ln \frac{r}{\alpha} - \sum_{n=1}^{k} \left[r^n - \left(\frac{\alpha^2}{r} \right)^n \right] Q_n \cos n\varphi = 0, \quad (7)$$

$$Q_0 \sin \varphi + \sum_{n=1}^{k} \left[r^n \sin (n-1) \varphi - \left(\frac{\alpha^2}{r} \right)^n \sin (n+1)\varphi \right] n Q_n = 0. \quad (8)$$

We note that relation (7) is satisfied at points on the straight sides BA and DC, while relation (8) is satisfied at points on AD and CB (Fig. 1). By virtue of symmetry with respect to the x-axis, it is sufficient in this problem to satisfy condition (7) at points on segments SA and DL only and condition (8) at points on segment AD only. If the centerline of the holes is not offset ($e = 0$), then the temperature distribution will be symmetrical with respect to the y-axis and, therefore, all odd Q_n coefficients will drop out. In that case conditions (7) and (8) need to be satisfied for φ only, which varies over the range $0 \le \varphi \le \pi/2$.

It does not appear possible to exactly satisfy conditions (7) and (8) at all points on the outer contour by a proper choice of constants Q_0 and Q_n in formula (6). We will satisfy these conditions approximately-exactly. Let us divide segments SA, AD, and DL into m equal parts. Inserting coordinates r_i, φ_i of the i-th dividing point into condition (7) if this point lies on segments SA or DL, or into condition (8) if this point lies on segment AD, we obtain an algebraic system of equations. The unknown quantities in this system are coefficients Q_0, Q_n ($n = 1, 2, \ldots, k$).

The number of dividing points can be made equal to the number of unknown coefficients in function (6). Then conditions (7) and (8) yield an algebraic system of more equations than is the number of unknown coefficients Q_0, Q_n. This method has the following drawbacks: a) in order to ensure an adequate degree of accuracy, it is necessary to solve a system of equations with many unknowns; b) function (6), which characterizes the temperature distribution in a perforated strip, has an unwieldy form with many terms.

For the solution of such practical problems as determining the stress distribution in heated bodies, it would be more convenient to deal with the temperature function expressed in a simpler form. We will use the method of "least squares" for determining the temperature distribution within a prescribed accuracy and with a minimum number of Q_n coefficients. The number of dividing points on segments SA, AD, Dl is made larger than the number of unknown coefficients. Satisfying the conditions (7) and (8) pointwise

TABLE 2. Values of the Coefficients when the Centerline of the Holes is Offset from the Centerline of the Strip (i.e., α = 0.3, γ = 1)

β	Q_0	Q_1	Q_2	$Q_3 \cdot 10$	$Q_4 \cdot 10^2$	$Q_5 \cdot 10^2$
0,10	0,6235	0,1111	0,2702	0,1054	—0,8502	0,3958
0,15	0,6329	0,1698	0,2777	0,1693	—0,6160	0,6498
0,20	0,6466	0,2327	0,2891	0,2492	—0,2353	0,9764
0,25	0,6659	0,3031	0,3080	0,3807	0,6297	1,5080
0,30	0,6911	0,3828	0,3335	0,5537	1,7471	2,2317
0,35	0,7243	0,4770	0,3709	0,8205	3,6634	3,5464
0,40	0,7682	0,5938	0,4285	1,2854	7,4963	6,5923

β	$Q_6 \cdot 10^2$	$Q_7 \cdot 10^3$	$Q_8 \cdot 10^3$	$Q_9 \cdot 10^3$	$Q_{10} \cdot 10^3$	$Q_{11} \cdot 10^3$
0,10	0,6482	0,9326	—	—	—	—
0,15	0,7345	1,4031	—	—	—	—
0,20	0,8596	1,8820	—	—	—	—
0,25	1,1529	3,2819	0,4283	—	—	—
0,30	1,5155	4,6723	0,6965	—	—	—
0,35	2,2659	8,2522	2,2534	0,5720	0,1516	—
0,40	4,3741	21,4893	9,5221	4,1660	1,4059	0,2689

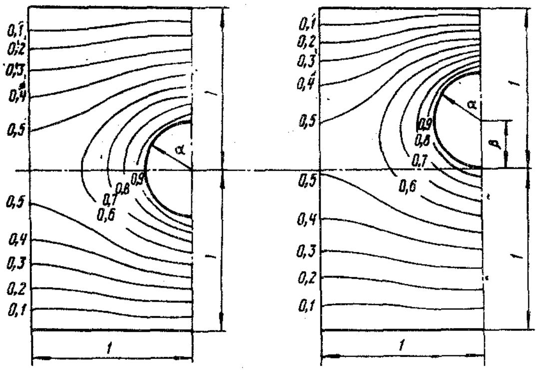

Fig. 2. Temperature distribution (θ = T/M) in a strip without offset (β = 0) and with offset (β = 0.3) of the centerline of the holes; relative hole radius α = 0.3.

will yield a system of more linear algebraic equations than is the number of unknown coefficients Q_0, Q_n. These equations of the system are called conditional equations [2]. The method of least squares makes it feasible to reduce the conditional equations to normal ones; the number of normal equations is equal to the number of unknowns. The solution to the system of normal equations yields the most probable values of the sought coefficients Q_0, Q_n in the temperature function (6).

The coefficients are determined as functions of the hole radius and of the offset. Values of these coefficients are listed in Table 1 for various lengths of the hole radius and for γ = 1 with a zero offset. The necessary accuracy for α = 0.05-0.45 is ensured by retaining four terms of the series in (6). As the radius increases, the number of terms increases and seven are necessary when α = 0.85.

The temperature distribution in a strip is shown in Fig. 2, with the centerline of the holes coinciding with the centerline of the strip (β = 0) or offset from it (β = 0.3). The dimensionless radius of the holes here is α = 0.3.

The calculations were made on a Minsk-22 computer. The accuracy of the solution was estimated on the basis of the error in satisfying the boundary conditions. The number of terms in expression (6) has been selected so as to keep the error within 1%. The temperature at the hole edge was regarded as 100%.

NOTATION

a	is the strip half-width;
b	is half the center-to-center distance between holes;
R	is the radius of the holes;
e	is the offset between the centerline of the holes and the centerline of the strip;
T	is the temperature;
M	is the temperature at the hole edges;
x and y	are the rectangular coordinates;
r and φ	are the polar coordinates;
φ_1 and φ_2	are angles;
θ, α, β, and γ	are parameters;
P_0, Q_0, P_n, and Q_n	are constant coefficients.

LITERATURE CITED

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